UNIT - 2

TWO DIMENSIONAL RANDOM VARIABLES

Joint distributions - Marginal and conditional distributions - Covariance - Correlation and Linear Regression - Transformation of random variables. Central limit theorem (for independent and identically distributed random variables)

2.0 Introduction

In unit 1, various aspects of the theory of single random variable were studied. The random variable was found to be a powerful concept. It enabled many realistic problems to be described in a probabilistic way such that practical measures could be applied to the problem even though it was random. From knowledge of the probability distribution or density function of impact position, we can solve for such practical measures as the mean value of impact position, its variance, and skew. These measures are not, however, a complete enough description of the problem in most cases. It may be necessary to extend the theory to include several random variables. Fortunately, many situations of our engineering problems are handled by the theory of two random variables. Hence, such important concepts as auto correlation, cross-correlation and covariance functions, which apply to random processes, are based on two random variables.

1. Two-dimensional random variable : Let S be the sample space. Let X = X (s) and Y = Y (s) be two functions each assigning a real number to each outcome $s \in S$. Then (X, Y) is a two-dimensional random variable.

2. Two-dimensional discrete random variables : If the possible values of (X, Y) are finite or countably infinite, then (X, Y) is called a two-dimensional discrete random variable. When (X, Y) is a two-dimensional discrete random variable the possible values of (X, Y) may be represented as (x_i, y_j), $i = 1, 2, \ldots n$, $j = 1, 2, 3, \ldots m$.

3. Two-dimensional continuous random variables : If (X, Y) can assume all values in a specified region R in the XY plane (X, Y) is called a two-dimensional continuous random variable.

2.1 Joint distributions - marginal and conditional distributions.

1. Joint probability distribution ■

The probabilities of the two events A = $\{X \le x\}$ and B = $\{Y \le y\}$ defined as functions of x and y, respectively, are called probability distribution functions.

$$F_X(x) = P(X \le x) \; ; \qquad F_Y(y) = P(Y \le y)$$

PROBLEMS UNDER DISCRETE RANDOM VARIABLES :

Example 2.1.1

From, the following table for bivariate distribution of (X, Y) find
(i) $P(X \leq 1)$, (ii) $P(Y \leq 3)$, (iii) $P(X \leq 1, Y \leq 3)$,
(iv) $P(X \leq 1/Y \leq 3)$, (v) $P(Y \leq 3/X \leq 1)$, (vi) $P(X + Y \leq 4)$.
(vii) The marginal distribution of X or Marginal PMF of X
(viii) The marginal distribution of Y or Marginal PMF of Y
(ix) The conditional distribution of X given Y = 2
(x) Examine X and Y are independent. (xi) E[Y − 2X]

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution :

X \ Y	1	2	3	4	5	6	$P_X(x) = f(x)$
0	0 P(0,1)	0 P(0,2)	$\frac{1}{32}$ P(0,3)	$\frac{2}{32}$ P(0,4)	$\frac{2}{32}$ P(0,5)	$\frac{3}{32}$ P(0,6)	$P(X=0) = \frac{8}{32}$ (sum of the I row)
1	$\frac{1}{16}$ P(1,1)	$\frac{1}{16}$ P(1,2)	$\frac{1}{8}$ P(1,3)	$\frac{1}{8}$ P(1,4)	$\frac{1}{8}$ P(1,5)	$\frac{1}{8}$ P(1,6)	$P(X=1) = \frac{20}{32}$ (sum of the II row)
2	$\frac{1}{32}$ P(2,1)	$\frac{1}{32}$ P(2,2)	$\frac{1}{64}$ P(2,3)	$\frac{1}{64}$ P(2,4)	0 P(2,5)	$\frac{2}{64}$ P(2,6)	$P(X=2) = \frac{4}{32}$ (sum of the III row)
$P_Y(y) = f(y)$	$P(Y=1) = \frac{3}{32}$ sum of the I column	$P(Y=2) = \frac{3}{32}$ sum of the II column	$P(Y=3) = \frac{11}{64}$ sum of the III column	$P(Y=4) = \frac{13}{64}$ sum of the IV column	$P(Y=5) = \frac{6}{32}$ sum of the V column	$P(Y=6) = \frac{16}{64}$ sum of the VI column	1

Y \ X	2	4	$P_Y(y) = P_{*j}$
1	0.10 P[2, 1]	0.15 P[4, 1]	P[Y=1] = 0.25 sum of the I row
3	0.20 P[2, 3]	0.30 P[4, 3]	P[Y=3] = 0.5 sum of the II row
5	0.10 P[2, 5]	0.15 P[4, 5]	P[Y=5] = 0.25 sum of the III row
$P_X(x) = P_{i*}$	P[X=2]=0.40 sum of the I column	P[X=4]=0.60 sum of the II column	1

If X and Y are independent, then

$$P[X = i] \times P[Y = j] = P[i, j], \quad \forall\, i, j$$

Here, $P[X = 2] \times P[Y = 1] = P[2, 1]$

i.e., $0.40 \times 0.25 = 0.1$

Similarly,

$$P[X = 2] \times P[Y = 3] = P[2, 3]$$

$$P[X = 2] \times P[Y = 5] = P[2, 5]$$

$$P[X = 4] \times P[Y = 1] = P[4, 1]$$

$$P[X = 4] \times P[Y = 3] = P[4, 3]$$

$$P[X = 4] \times P[Y = 5] = P[4, 5]$$

$\therefore$ X and Y are independent

Example 2.1.23

The joint p.d.f of the random variables X and Y is given by $P(x, y) = xe^{-x(y+1)}$ where $0 \le x, y < \infty$. (i) Find P(x) and P(y) and (ii) Are the random variables independent ?

Solution :

$$P(x) = \int_{-\infty}^{\infty} P(x,y)\,dy = \int_{0}^{\infty} xe^{-x(y+1)}\,dy$$

$$= \left[-\frac{xe^{-x(y+1)}}{x}\right]_0^{\infty}$$

$$= e^{-x},\ 0 \le x < \infty$$

$$P(y) = \int_{-\infty}^{\infty} P(x,y)\,dx = \int_{0}^{\infty} xe^{-x(y+1)}\,dx$$

$$u = x\ ;\ v = e^{-x(y+1)}$$

$$u' = 1\ ;\ v_1 = -e^{-x(y+1)} / y+1$$

$$u'' = 0\ ;\ v_2 = e^{-x(y+1)} / (y+1)^2$$

$$P(y) = \left[-\frac{x}{y+1}e^{-x(y+1)} - \frac{1}{(y+1)^2}e^{-x(y+1)}\right]_0^{\infty}$$

$$= \frac{1}{(y+1)^2},\ 0 \le y < \infty$$

Consider $P(x) \,.\, P(y) = e^{-x}\dfrac{1}{(y+1)^2} \neq P(x,y)$

$\Rightarrow$ X and Y are not independent.

2.2 COVARIANCE, CORRELATION AND REGRESSION

When two or more random variables are defined on a probability space, it is useful to describe how they vary together, that is, it is useful to measure the relationship between the variables. A common measure of the relationship between two random variables is the covariance. To define the covariance, we need to describe the expected value of a function of two random variables $h(x, y)$. The definition simply extends that used for a function of a single random variable.

1. COVARIANCE

If X and Y are random variables, then co-variance between them is defined as,

[A.U N/D 2019 (R17)]

$$\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))]$$

$$= E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$

$$= E[XY] - E(X)E(Y) - E(X)E(Y) + E(X)E(Y)$$

$$= E(XY) - E(X)E(Y)$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

Note : If X and Y are independent, then $\text{Cov}(X, Y) = 0$

If X and Y are independent, then $E[XY] = E(X) . E(Y)$

$\Rightarrow \text{Cov}(X, Y) = 0$

(i) Cov (aX, bY) = ab Cov (X, Y)

$$\text{Cov}(aX, bY) = E[(aX)(bY)] - E(aX)E(bY)$$

$$= ab\, E(XY) - ab\, E(X)E(Y)$$

$$= ab\,[E(XY) - E(X)E(Y)] = ab\, \text{Cov}(X, Y)$$

(ii) Cov (X + a, Y + b) = Cov (X, Y)

$$\text{Cov}(X + a, Y + b) = E[(X + a)(Y + b)] - E(X + a)E(Y + b)$$

$$= E[XY + bX + aY + ab] - [E(X) + a][E(Y) + b]$$

$$= E(XY) + b\,E(X) + a\,E(Y) + ab - E(X)E(Y) - aE(Y) - b\,E(X) - ab$$

$$= E(XY) - E(X)E(Y) = \text{Cov}(X, Y)$$

(iii) Cov (aX + b, cY + d) = ac Cov (X, Y)

$$\text{Cov}(aX + b, cY + d) = E[(aX + b)(cY + d)] - E(aX + b)E(cY + d)$$

$$= E[ac\,XY + ad\,X + bc\,Y + bd] - [aE(X) + b][cE(Y) + d]$$

$$= ac\,E(XY) + ad\,E(X) + bc\,E(Y) + bd - ac\,E(X)E(Y) - ad\,E(X) - bc\,E(Y) - bd$$

$$= ac\,[E(XY) - E(X)E(Y)] = ac\,\text{Cov}(X, Y)$$

Example 2.2.3

Ten participants were ranked according to their performance in a mustical test by the 3 Judges in the following data.

	1	2	3	4	5	6	7	8	9	10
Rank by X	1	6	5	10	3	2	4	9	7	8
Rank by Y	3	5	8	4	7	10	2	1	6	9
Rank by Z	6	4	9	8	1	2	3	10	5	7

Using rank correlation method, discuss which pair of judges has the nearest approach to common likings of music.

Solution :

x_i	y_i	z_i	$d_1=x_i-y_i$	$d_2=y_i-z_i$	$d_3=x_i-z_i$	d_1^2	d_2^2	d_3^2
1	3	6	−2	−3	−5	4	9	25
6	5	4	1	1	2	1	1	4
5	8	9	−3	−1	−4	9	1	16
10	4	8	6	−4	2	36	16	4
3	7	1	−4	6	2	16	36	4
2	10	2	−8	8	0	64	64	0
4	2	3	2	−1	1	4	1	1
9	1	10	8	−9	−1	64	81	1
7	6	5	1	1	2	1	1	4
8	9	7	−1	2	1	1	4	1

The rank correlation co-efficient between X and Y is given by

$$r(X, Y) = 1 - \frac{6\Sigma d_1^2}{n(n^2-1)} \qquad \text{Here, } n = 10$$

$$= 1 - \frac{6 \times 200}{10(100-1)} = -0.212$$

The rank correlation co-efficient between Y and Z is given by

$$r(Y, Z) = 1 - \frac{6\Sigma d_2^2}{n(n^2-1)} = 1 - \frac{6 \times 214}{10(100-1)} = -0.296$$

The rank correlation co-efficient between X and Z is given by

$$r(X, Z) = 1 - \frac{6\Sigma d_3^2}{n(n^2-1)} = 1 - \frac{6 \times 60}{10(100-1)} = 0.636$$

Since the rank correlation coefficient between X and Z is positive and maximum, we conclude that the pair of judges X and Z has the nearest approach to common liking in music.

2.3 REGRESSION

■ (1) Regression ■

Regression is a mathematical measure of the average relationship between two or more variables in terms of the original limits of the data.

■ (2) Lines of regression ■

(1) The line of regression of y on x is given by

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \quad \text{... (1)}$$

(2) The line of regression of x on y is given by

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \quad \text{... (2)}$$

Note : Both the lines of regression passes through $(\bar{X}, \bar{Y})$

■ (3) Regression coefficients ■

(1) Regression coefficient of y on x is $r\frac{\sigma_y}{\sigma_x} = b_{yx}$

(2) Regression coefficient of x on y is $r\frac{\sigma_x}{\sigma_y} = b_{xy}$

Correlation coefficient $r = \pm\sqrt{b_{yx}\, b_{xy}}$

where $b_{yx} = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(x - \bar{x})^2}$; $b_{xy} = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(y - \bar{y})^2}$

Example 2.3.2

The two lines of regression are

$$8x - 10y + 66 = 0 \quad \ldots (A)$$

$$40x - 18y - 214 = 0 \quad \ldots (B)$$

The variance of x is 9. Find (i) The mean values of x and y (ii) Correlation co-efficient between x and y [AU N/D 2008]

[A.U CBT M/J 2010, CBT N/D 2011, CBT A/M 2011]

[A.U A/M 2015 (RP) R13]

[A.U M/J 2015 R13 PQT] [A.U M/J 2016 R13 RP]

Solution: (i) Since both the lines of regression passes through the mean values $\bar{x}$ and $\bar{y}$, the point $(\bar{x}, \bar{y})$ must satisfy the two given regression lines

$$8\bar{x} - 10\bar{y} = -66 \quad \ldots (1)$$

$$40\bar{x} - 18\bar{y} = 214 \quad \ldots (2)$$

$$(1) \times 5 \Rightarrow \quad 40\bar{x} - 50\bar{y} = -330 \quad \ldots (3)$$

$$(2) \times 1 \Rightarrow \quad 40\bar{x} - 18\bar{y} = 214 \quad \ldots (4)$$

$$(3) - (4) \Rightarrow \quad -32\bar{y} = -544$$

$$\bar{y} = 17$$

Sub in (1) we get

$$8\bar{x} - 10(17) = -66$$

$$\bar{x} = 13$$

Hence the mean values are given by $\bar{x} = 13$, $\bar{y} = 17$

(ii)

(A) $\Rightarrow 8x = 10y - 66$ $\Rightarrow x = \frac{10}{8}y - \frac{66}{8}$ i.e., $b_{xy} = \frac{10}{8}$	(B) $\Rightarrow 18y = 40x - 214$ $\Rightarrow y = \frac{40}{18}x - \frac{214}{18}$ i.e., $b_{yx} = \frac{40}{18}$	$r^2 = b_{xy} b_{yx}$ $= \left(\frac{10}{8}\right)\left(\frac{40}{18}\right)$ $= 2.77$ $r = 1.66 \nless 1$

(A) $\Rightarrow 10y = 8x + 66$ $\Rightarrow y = \frac{8}{10}x + \frac{66}{10}$ i.e., $b_{yx} = \frac{8}{10}$	(B) $\Rightarrow 40x = 18y + 214$ $\Rightarrow x = \frac{18}{40}y + \frac{214}{40}$ i.e., $b_{xy} = \frac{18}{40}$	$r^2 = b_{yx}\, b_{xy}$ $= \left(\frac{8}{10}\right)\left(\frac{18}{40}\right)$ $= 0.36$ $r = \pm 0.6$

Since both the regression coefficients are positive r must be positive $r = 0.6$.

Example 3.3.3

The following table gives according to age x, the frequency of marks obtained 'y' by 100 students in an intelligence test. Measure the degree of relationship between age and intelligence test.

Age/marks	16-17	17-18	18-19	19-20
30-40	20	10	3	2
40-50	4	28	6	4
50-60	0	5	11	0
60-70	0	0	2	0
70-80	0	0	0	5

The origin is taken as $\bar{x} = 18.5$ and $\bar{y} = 55$

$$X = \frac{x-\bar{x}}{h} = \frac{x-18.5}{1} \quad [\because h = \text{difference between } x \text{ values}]$$

$$Y = \frac{y-\bar{y}}{k} = \frac{y-55}{10} \quad [\because k = \text{difference between } y \text{ values}]$$

$f_y \rightarrow$ sum of the each row

$f_x \rightarrow$ sum of the each column

$f_{xy} \rightarrow$ Given frequency

$N = \Sigma f_x = \Sigma f_y = 100$

Y \ X	16.5	17.5	18.5	19.5	f_y	Y	Y f_y	$Y^2 f_y$	XYf_{xy}
35	$f_{xy}=20$ $XY=4$ $XYf_{xy}=80$	10 2 $=20$	3 0 $=0$	2 -2 $=-4$	35	-2	-70	140	96
45	4 2 $=8$	28 1 $=28$	6 0 $=0$	4 -1 $=-4$	42	-1	-42	42	32
55	0 0 $=0$	5 0 $=0$	11 0 $=0$	0 0 $=0$	16	0	0	0	0
65	0 -2 $=0$	0 -1 $=0$	2 0 $=0$	0 1 $=0$	2	1	2	2	0
75	0 -4 $=0$	0 -2 $=0$	0 0 $=0$	5 2 $=10$	5	2	10	20	10
f_x	24	43	22	11	100	0	-100	204	138
X	-2	-1	0	1	-2				
Xf_x	-48	-43	0	11	-80				
$X^2 f_x$	96	43	0	11	150				
XYf_{xy}	88	48	0	2	138				

In each cell upper values are f_{xy} (given), middle are XY, lower are XYf_{xy}

$$\sigma_X^2 = \frac{\Sigma (X^2 f_x)}{N} - \left(\frac{\Sigma (X f_x)}{N}\right)^2 = \frac{150}{100} - \left(\frac{-80}{100}\right)^2 = 0.86 \therefore \sigma_X = 0.927$$

$$\sigma_Y^2 = \frac{\Sigma (Y^2 f_y)}{N} - \left(\frac{\Sigma (Y f_y)}{N}\right)^2 = \frac{204}{100} - \left(\frac{-100}{100}\right)^2 = 1.04 \Rightarrow \sigma_Y = 1.019$$

$$\rho = \frac{\Sigma (XY f_{xy})}{N} - \left(\frac{\Sigma (X f_x)}{N}\right)\left(\frac{\Sigma (Y f_y)}{N}\right) = \frac{138}{100} - \left(\frac{-80}{100}\right)\left(\frac{-100}{100}\right)$$

$$= 1.38 - 0.8 = 0.58$$

$$\therefore r = \frac{\rho}{\sigma_X \sigma_Y} = \frac{0.58}{(0.927)(1.019)} = 0.6137$$

2.4 TRANSFORMATION OF RANDOM VARIABLES

■ 1. Two functions of two random variables ■

If (X, Y) is a two dimensional random variable with joint p.d.f $f_{XY}(x,y)$ and if $Z = g(X, Y)$ and $W = h(X, Y)$ are two other random variables then the joint p.d.f of (Z, W) is given by,

$$f_{ZW}(z,w) = \frac{f_{XY}(x,y)}{|J|} \quad \text{where } J = \frac{\partial(z,w)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix}$$

(or) $$f_{ZW}(z,w) = |J|\, f_{XY}(x,y) \quad \text{where } J = \frac{\partial(x,y)}{\partial(z,w)}$$

Note : This result holds good, only if the equation $Z = g(X, Y)$ and $W = h(X, Y)$ when solved, give unique values of x and y in terms of z and w.

■ 2. One function of two random variables ■

If a random variable Z is defined as $Z = g(X, Y)$, where X and Y are given random variables with joint p.d.f $f(x,y)$. To find the pdf of Z, we introduce a second Random variable $W = h(X, Y)$ and obtain the joint p.d.f of (Z, W), by using the previous result. Let it be $f_{ZW}(z,w)$. The required p.d.f of Z is then obtained as the marginal p.d.f is $f_Z(z)$ is obtained by simply integrating $f_{ZW}(Z, W)$ w.r. to w.

$$\text{i.e., } f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\, dw$$

Example 2.4.1

Let (X, Y) be a two-dimensional non-negative continuous random variable having the joint density.

$$f(x, y) = \begin{cases} 4xy\, e^{-(x^2+y^2)}, & x \ge 0,\ y \ge 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the density function of $U = \sqrt{X^2 + Y^2}$

[A.U A/M 2005] [A.U Tvli M/J 2010. Tvli A/M 2011] [A.U M/J 2016 R13 (RP)]

Solution:

The density function of U is

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\, dv \quad \dots (1)$$

$$f(u,v) = |J|\, f(x,y) \quad \dots (2)$$

$$\text{where } J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \dots (3)$$

Given :

$u = \sqrt{x^2+y^2}$ ...(a) $\quad v = x$ ... (b)

$u^2 = x^2+y^2 \quad v^2 = x^2$

$y^2 = u^2 - x^2 = u^2 - v^2$

$\therefore x = v \quad y = \sqrt{u^2-v^2}$ ... (c)

$\frac{\partial x}{\partial u} = 0 \quad \frac{\partial y}{\partial u} = \frac{u}{\sqrt{u^2-v^2}}$

$\frac{\partial x}{\partial v} = 1 \quad \frac{\partial y}{\partial v} = \frac{-v}{\sqrt{u^2-v^2}}$

$$(3) \Rightarrow J = \begin{vmatrix} 0 & 1 \\ \frac{u}{\sqrt{u^2-v^2}} & \frac{-v}{\sqrt{u^2-v^2}} \end{vmatrix} = \frac{-u}{\sqrt{u^2-v^2}}$$

$$\therefore |J| = \frac{u}{\sqrt{u^2-v^2}}$$

$f(x,y) = 4xy\, e^{-(x^2+y^2)}$

$= 4v\sqrt{u^2-v^2}\, e^{-u^2}$ ... (d)

$$(2) \Rightarrow f(u,v) = \frac{u}{\sqrt{u^2-v^2}} \left[4v\sqrt{u^2-v^2}\, e^{-u^2}\right] = 4uv\, e^{-u^2}$$

$x \ge 0 \quad y \ge 0$

$(b) \Rightarrow v \ge 0 \quad (c) \Rightarrow \sqrt{u^2-v^2} \ge 0$

$\Rightarrow u^2 - v^2 \ge 0$

$\Rightarrow u^2 \ge v^2$

$\Rightarrow u \ge v$

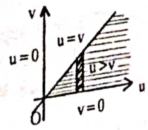
$$(1) \Rightarrow f(u) = \int_0^u 4uv\, e^{-u^2}\, dv$$

$$= 4u\, e^{-u^2} \int_0^u v\, dv$$

$$= 4u\, e^{-u^2} \left[\frac{v^2}{2}\right]_0^u$$

$$= 4u\, e^{-u^2} \left[\frac{u^2}{2} - 0\right]$$

$$= 2u^3 e^{-u^2},\ u \ge 0$$



Example 2.4.2

If U = X + Y and V = X − Y, how are the joint p.d.f's of (X, Y) and (U, V) related?

Solution :

$$f(u,v) = |J|\, f(x,y) \quad \ldots (1)$$

$$\text{where} \quad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \ldots (2)$$

Given : $u = x + y$ ... (3), $v = x - y$ ... (4)

(3) + (4) ⇒ $2x = u + v$

$$x = \frac{1}{2}(u+v)$$

$$\frac{\partial x}{\partial u} = \frac{1}{2}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}$$

(3) − (4) ⇒ $2y = u - v$

$$y = \frac{1}{2}(u-v)$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}$$

$$\frac{\partial y}{\partial v} = -\frac{1}{2}$$

$$(2) \Rightarrow \quad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\therefore |J| = \frac{1}{2}$$

$$(1) \Rightarrow f(u,v) = \frac{1}{2} f(x,y)$$

Example 2.4.3

If X and Y are independent random variables with p.d.f e^{-x}, x≥0; e^{-y}, y ≥ 0 respectively. Find the density function of $U = \frac{X}{X+Y}$ and V = X + Y. Are U & V independent ?

[A.U. N/D 2006] [A.U Trichy M/J 2009, A.U N/D 2009, N/D 2011, N/D 2013] [A.U N/D 2015 R13, N/D 2017 (RP) R13, A/M 2017 (RP) R13] [A.U N/D 2018 R-17 PS]

Solution :

Since, X and Y are independent,

$$f(x,y) = e^{-x} \cdot e^{-y} = e^{-(x+y)}, \ x, y \geq 0$$

The density function of U is

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\, dv \quad \ldots (1)$$

The density function of V is

$$f(v) = \int_{-\infty}^{\infty} f(u,v)\, du \quad \ldots (2)$$

where $f(u,v) = |J|\, f(x,y)$ ... (3)

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \ldots (4)$$

Given :

$u = \frac{x}{x+y}$ ...(a) | $v = x + y$ .. (b)

⇒ $u = \frac{x}{v}$ by (b) | $y = v - x$

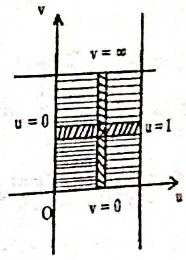
⇒ $x = uv$ ... (c) | $y = v - uv$ ...(d)

$\frac{\partial x}{\partial u} = v$ | $\frac{\partial y}{\partial u} = -v$

$\frac{\partial x}{\partial v} = u$ | $\frac{\partial y}{\partial v} = 1 - u$

$f(x,y) = e^{-(x+y)} = e^{-v}$

$x \geq 0$	$y \geq 0$
$uv \geq 0$ by (c)	$v - uv \geq 0$ by (d)
⇒ $v \geq 0$	⇒ $v \geq uv$
	⇒ $1 \geq u$
	⇒ $u \leq 1$



$$(4) \Rightarrow J = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v - uv + uv = v$$

$$(3) \Rightarrow f(u,v) = v e^{-v}$$

$$(1) \Rightarrow f(u) = \int_0^{\infty} v e^{-v}\, dv$$

$$= \left[v\left(\frac{e^{-v}}{-1}\right) - (1)\left(\frac{e^{-v}}{(-1)^2}\right) \right]_0^{\infty}$$

$$= \left[-v e^{-v} - e^{-v} \right]_0^{\infty}$$

$$= [(-0 - 0) - (-0 - 1)] = 1$$

$$(2) \Rightarrow f(v) = \int_0^1 v e^{-v}\, du = v e^{-v} \int_0^1 du$$

$$= v e^{-v} \left[u \right]_0^1 = v e^{-v} [1 - 0] = v e^{-v}, \ v \geq 0$$

Now, $f(u) f(v) = (1)(v e^{-v})$

$$= v e^{-v}$$

$$= f(u,v)$$

∴ u and v are independent random variables